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ON SHAKEDOWN OF SHAPE MEMORY ALLOYS WITH PERMANENT INELASTICITY

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Abstract. Shape memory alloys (SMAs) offer interesting perspectives in various fields such as aeronautics, robotics, biomedical sciences, or structural engineering. The distinctive properties of those materials stem from a solid/solid phase transformation occurring at a microscopic level. Modeling the rather complex behavior of SMAs is a topic of active research. Lately, SMA models coupling phase-transformation with permanent inelasticity have been proposed to capture degradation effects which are frequently observed experimentally for cyclic loadings. In this paper, the classical static and kinematic shakedown of plasticity theory are extended to such material models. Those results gives conditions for the energy dissipation to remain bounded, and might be relevant for the fatigue design of SMA systems.

1 INTRODUCTION

This communication is concerned with shakedown theorems for Shape Memory Alloys (SMAs). The peculiar properties of SMAs are the result of a solid/solid phase transformation between different crystallographic structures. Much effort has been devoted to developing constitutive laws for describing the behaviour of SMAs. The phase transformation is typically tracked by an internal variable $\boldsymbol{\alpha}_1$ which - depending on the complexity of the material model - may be scalar or vectorial [1, 6, 11, 12]. In most of SMA models, the internal variable $\boldsymbol{\alpha}_1$ must comply with some a priori inequalities that result from the mass conservation in the phase transformation process. The presence of such constraints constitutes a crucial difference with standard plasticity models, and calls for special attention when the structural evolution problem is considered [6, 16, 22, 21] . Non-smooth mechanics [5] offer a sound mathematical framework for handling constraints on state variables. The large-time behavior of solids in non-smooth mechanics has been addressed in [18]: a static shakedown theorem has been proposed, taking the form of a sufficient condition for the evolution to become elastic in the large-time limit. When the shakedown limit provided by that theorem is exceeded, it was found that the large-time behaviour is dependent on the initial state: in the case of cyclic loadings, some initial conditions lead to shakedown whereas some others lead to alternating phase transformation. Such a feature is not found in standard plasticity.

The shakedown theorem in [18] is path-independent - in the spirit of the original Melan theorem [14, 9] - and applies to a wide range of constitutive models of phase transformation in SMAs. Lately, models coupling phase-transformation and plasticity have been proposed in an effort to describe permanent inelasticy effects which are experimentally observed in SMAs [2, 10, 23, 3]: although phase transformation in SMAs is the main inelastic mechanism, dislocation motions also exist and are (partly) responsible for such effects as training and degradation in cyclic loadings. To model such a behavior, two internal variables are generally introduced: in addition to the (constrained) variable $\boldsymbol{\alpha}_1$ describing the phase transformation, an additional variable $\boldsymbol{\alpha}_2$ is used to describe permanent inelasticity. As discussed in [2, 23], it is essential to introduce a coupling term between those two variables in the free energy. Extending the approach used in [18], we present a static shakedown theorem for SMA models coupling phase-transformation and permanent inelasticity. For a parametrized loading history, that theorem gives a 'static' safety factor with respect to shakedown. Using min-max duality, a kinematic theorem and a corresponding 'kinematic' safety factor are introduced. Because of space limitations and so as not to obscure the presentation, we only sketch the proofs of the theorems.

2 CONSTITUTIVE LAWS

We first describe the class of constitutive models that we consider in this paper. The local state of the material is described by the (linearized) strain $\boldsymbol{\epsilon}$ and two internal variables $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ living respectively in vectorial spaces denoted by $\mathbb{A}_1$ and $\mathbb{A}_2$. The variable $\boldsymbol{\alpha}_1$ tracks the phase transformation, whereas the variable $\boldsymbol{\alpha}_2$ describes permanent inelasticity effects. Because of mass conservation in the phase transformation process, the variable $\boldsymbol{\alpha}_1$ is constrained to take values in a given bounded subset $\mathcal{T}_1$ of $\mathbb{A}_1$. The set $\mathcal{T}_1$ is assumed to be closed and convex in the following. Adopting the framework of generalized standard materials in non-smooth mechanics [8, 5], the behaviour of the material is determined by the free energy function $w(\boldsymbol{\epsilon}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$ and the dissipation potential $\Phi(\dot{\boldsymbol{\alpha}}_1, \dot{\boldsymbol{\alpha}}_2)$. More precisely, denoting by $\dot{\boldsymbol{\alpha}}_i$ the left-time derivative of $\boldsymbol{\alpha}_i$, the constitutive equations are

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \ , \ \boldsymbol{A}_i = -\frac{\partial w}{\partial \boldsymbol{\alpha}_i}(\boldsymbol{\epsilon}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2), \tag{1}$$

$$\begin{aligned} &\boldsymbol{A}_1 = \boldsymbol{A}_1^d + \boldsymbol{A}_1^r, \ \boldsymbol{A}_2 = \boldsymbol{A}_2^d, \\ &(\boldsymbol{A}_1^d, \boldsymbol{A}_2^d) \in \partial\Phi(\dot{\boldsymbol{\alpha}}_1, \dot{\boldsymbol{\alpha}}_2), \\ &\boldsymbol{A}_1^r \in \partial I_{\mathcal{T}_1}(\boldsymbol{\alpha}_1), \end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{A}_i$ is the thermodynamical force associated to $\boldsymbol{\alpha}_i$, and ∂ is the subdifferential operator [4]. We consider free energy functions $w(\boldsymbol{\epsilon}, \boldsymbol{\alpha})$ of the form

$$w(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{\epsilon} - \boldsymbol{K}_1.\boldsymbol{\alpha}_1 - \boldsymbol{K}_2.\boldsymbol{\alpha}_2) : \boldsymbol{L} : (\boldsymbol{\epsilon} - \boldsymbol{K}_1.\boldsymbol{\alpha}_1 - \boldsymbol{K}_2.\boldsymbol{\alpha}_2) + f(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) + h(\boldsymbol{\alpha}_1) \quad (3)$$

where $\boldsymbol{L}$ is a symmetric positive definite, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are linear operators, $f : \mathcal{T}_1 \times \mathbb{A}_2 \mapsto \mathbb{R}_+$ is a convex differentiable function, $h : \mathcal{T}_1 \mapsto \mathbb{R}_+$ is differentiable (but not necessarily convex). With the form (3) of the free energy, the relation (1) becomes

$$\begin{array}{l} \boldsymbol{\sigma} = \boldsymbol{L} : (\boldsymbol{\epsilon} - \boldsymbol{K}_1.\boldsymbol{\alpha}_1 - \boldsymbol{K}_2.\boldsymbol{\alpha}_2) \ , \\ \boldsymbol{A}_1 = {}^t\boldsymbol{K}_1 : \boldsymbol{\sigma} - f_{,1}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) - h'(\boldsymbol{\alpha}_1) \ , \\ \boldsymbol{A}_2 = {}^t\boldsymbol{K}_2 : \boldsymbol{\sigma} - f_{,2}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \end{array} \quad (4)$$

where ${}^t\boldsymbol{K}_i$ is the transpose of $\boldsymbol{K}_i$ and $f_{,i}$ is the partial derivative of f with respect to $\boldsymbol{\alpha}_i$.

It is convenient to use the compact notations $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$, $\boldsymbol{A}^d = (\boldsymbol{A}_1^d, \boldsymbol{A}_2^d)$, $\boldsymbol{A}^r = (\boldsymbol{A}_1^r, 0)$. The gradient of the function f (resp. h) with respect to $\boldsymbol{\alpha}$ is denoted by ∇f (resp. ∇h), i.e $\nabla f = (f_{,1}, f_{,2})$ and $\nabla h = (h'(\boldsymbol{\alpha}_1), 0)$. We also introduce the linear operator $\boldsymbol{K}$ defined on $\mathbb{A}_1 \times \mathbb{A}_2$ by the relation $\boldsymbol{K}.\boldsymbol{\alpha} = \boldsymbol{K}_1.\boldsymbol{\alpha}_1 + \boldsymbol{K}_2.\boldsymbol{\alpha}_2$. Eq. (4) can be rewritten in the compact form

$$\begin{array}{l} \boldsymbol{\sigma} = \boldsymbol{L} : (\boldsymbol{\epsilon} - \boldsymbol{K}.\boldsymbol{\alpha}) \ , \\ \boldsymbol{A} = {}^t\boldsymbol{K} : \boldsymbol{\sigma} - \nabla f(\boldsymbol{\alpha}) - \nabla h(\boldsymbol{\alpha}). \end{array} \quad (5)$$

As usual in the framework of standard generalized materials, the dissipation potential Φ is assumed to be convex, positive, null at the origin. Those standard requirements ensure the positiveness of the dissipation, in compliance with the second principle of thermodynamics. The convex set $\mathcal{C} = \partial\Phi(0)$ can be interpreted as the elasticity domain of the material.

As stated in the introduction, the class of constitutive models considered in this paper is motivated by recently proposed models of phase transformation coupled with permanent inelasticity. As an illustrative example, we consider the model proposed in [2]. With the present set of notations, the internal variable $\boldsymbol{\alpha}_1$ in that model is a deviatoric second-order tensor submitted to the restriction $\|\boldsymbol{\alpha}_1\| \leq \alpha^T$ where α^T is a material parameter. The internal variable $\boldsymbol{\alpha}_2$ is a deviatoric second-order tensor that can take any value. The free energy w is of the form (3) with $h = 0$, $\boldsymbol{K}_2 = 0$, $\boldsymbol{K}_1 = \boldsymbol{I}$, and

$$f(\boldsymbol{\alpha}) = \beta\|\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2\| + \frac{1}{2}a_1\|\boldsymbol{\alpha}_1\|^2 + \frac{1}{2}a_2\|\boldsymbol{\alpha}_2\|^2 - b\,\boldsymbol{\alpha}_1.\boldsymbol{\alpha}_2.$$

In that last expression, β, a_1, a_2 and b are all material parameters. In particular, β is non-negative in the super elastic regime, i.e. for sufficiently high temperature. In such condition, the function f is convex provided that

$$a_1 + a_2 \geq 0 \ , \ a_1 a_2 - b^2 \geq 0. \quad (6)$$

Typical values used in [2] are $a_1 = 10^3$ MPa, $a_2 = 1.5\,10^4$ MPa, $b = 2\,10^3$ MPa, which satisfy (6). The elasticity domain $\mathcal{C}$ in [2] consists of pairs $(\boldsymbol{A}_1, \boldsymbol{A}_2)$ of deviatoric tensors verifying

$$\|\boldsymbol{A}_1\| + \kappa\|\boldsymbol{A}_2\| \leq R \tag{7}$$

where κ and R are non-negative material parameters.

3 QUASI-STATIC EVOLUTION OF A CONTINUUM

We now consider a continuum submitted to a prescribed loading history. The continuum occupies a domain Ω and is submitted to body forces $\boldsymbol{f}^d$. Displacements $\boldsymbol{u}^d$ are imposed on a part Γ_u of the boundary Γ, and tractions $\boldsymbol{T}^d$ are prescribed on $\Gamma_T = \Gamma - \Gamma_u$. The given functions $\boldsymbol{f}^d$,$\boldsymbol{u}^d$,$\boldsymbol{T}^d$ depend on position $\boldsymbol{x}$ and time t. The stress and state variables $(\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{\alpha})$ in the continuum are also expected to depend on $(\boldsymbol{x}, t)$. In order to alleviate the expressions, this dependence will be omitted in the notations, unless in the case of possible ambiguities.

Quasi-static evolutions of the continuum are governed by the following system:

$$\begin{gathered} \boldsymbol{\sigma} \in \mathcal{K}_\sigma \ , \ \boldsymbol{\epsilon} \in \mathcal{K}_\epsilon \ , \ \boldsymbol{\alpha} \in \mathcal{T}, \\ \boldsymbol{A}^d \in \partial\Phi(\dot{\boldsymbol{\alpha}}) \ , \ \boldsymbol{A}^r \in \partial I_{\mathcal{T}}(\boldsymbol{\alpha}), \\ \boldsymbol{\sigma} = \boldsymbol{L} : (\boldsymbol{\epsilon} - \boldsymbol{K}.\boldsymbol{\alpha}), \\ {}^t\boldsymbol{K} : \boldsymbol{\sigma} - \nabla f(\boldsymbol{\alpha}) - \nabla h(\boldsymbol{\alpha}) = \boldsymbol{A}^d + \boldsymbol{A}^r, \end{gathered} \tag{8}$$

where $\mathcal{K}_\sigma$ and $\mathcal{K}_\epsilon$ are respectively the sets of admissible stress and strain fields, defined by $\mathcal{K}_\sigma = \{\sigma : \operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f}^d = 0 \text{ in } \Omega; \boldsymbol{\sigma}.\boldsymbol{n} = \boldsymbol{T}^d \text{ on } \Gamma_T\}$ and $\mathcal{K}_\epsilon = \{\boldsymbol{\epsilon} : \boldsymbol{\epsilon} = (\nabla\boldsymbol{u} + {}^t\nabla\boldsymbol{u})/2 \text{ in } \Omega;\ \boldsymbol{u} = \boldsymbol{u}^d \text{ on } \Gamma_u\}$. The set $\mathcal{T}$ in (8) is the subset of $\mathbb{A}_1 \times \mathbb{A}_2$ defined by $\mathcal{T} = \mathcal{T}_1 \times \mathbb{A}_2$. Note that any $\boldsymbol{A}^r \in \partial I_{\mathcal{T}}(\dot{\boldsymbol{\alpha}})$ is of the form $\boldsymbol{A}^r = (\boldsymbol{A}_1^r, 0)$ with $\boldsymbol{A}_1^r \in \partial I_{\mathcal{T}_1}(\dot{\boldsymbol{\alpha}}_1)$.

We introduce the so-called *fictitious elastic response* $(\boldsymbol{\sigma}^E, \boldsymbol{\epsilon}^E)$ of the continuum, defined by

$$\boldsymbol{\sigma}^E \in \mathcal{K}_\sigma \ , \boldsymbol{\epsilon}^E \in \mathcal{K}_\epsilon \ , \boldsymbol{\sigma}^E = \boldsymbol{L} : \boldsymbol{\epsilon}^E. \tag{9}$$

Setting $\boldsymbol{\rho} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^E$ and noting that $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^E + \boldsymbol{L}^{-1} : \boldsymbol{\rho} + \boldsymbol{K}.\boldsymbol{\alpha}$, the system (8) can be recast as

$$\begin{gathered} \boldsymbol{\rho} \in \mathcal{K}_\sigma^0 \ , \ \boldsymbol{\alpha} \in \mathcal{T}, \\ \boldsymbol{A}^d \in \partial\Phi(\dot{\boldsymbol{\alpha}}) \ , \ \boldsymbol{A}^r \in \partial I_{\mathcal{T}}(\boldsymbol{\alpha}), \\ \boldsymbol{L}^{-1} : \boldsymbol{\rho} + \boldsymbol{K}.\boldsymbol{\alpha} \in \mathcal{K}_\epsilon^0, \\ {}^t\boldsymbol{K} : (\boldsymbol{\sigma}^E + \boldsymbol{\rho}) - \nabla f(\boldsymbol{\alpha}) - \nabla h(\boldsymbol{\alpha}) = \boldsymbol{A}^d + \boldsymbol{A}^r. \end{gathered} \tag{10}$$

The sets $\mathcal{K}_\sigma^0$ and $\mathcal{K}_\epsilon^0$ in (10) are defined by

$$\begin{aligned} \mathcal{K}_\sigma^0 &= \{\boldsymbol{\sigma} : \operatorname{div}\boldsymbol{\sigma} = 0 \text{ in } \Omega; \boldsymbol{\sigma}.\boldsymbol{n} = 0 \text{ on } \Gamma_T\}, \\ \mathcal{K}_\epsilon^0 &= \{\boldsymbol{\epsilon} : \boldsymbol{\epsilon} = (\nabla\boldsymbol{u} +^t \nabla\boldsymbol{u})/2 \text{ in } \Omega;\ \boldsymbol{u} = 0 \text{ on } \Gamma_u\}. \end{aligned} \tag{11}$$

In the following, we examine conditions under which the energy dissipation $\int_0^T \int_\Omega \boldsymbol{A}^d.\dot{\boldsymbol{\alpha}} d\boldsymbol{x} dt$ remains bounded (with respect to time T) for all solutions of the evolution problem (10) (or equivalently (8)). Such a situation is refered to as shakedown.

4 STATIC SHAKEDOWN THEOREM

Assume there exists $m > 1$, $T > 0$ and time-independent fields $(\boldsymbol{\rho}_*, \boldsymbol{\alpha}_*, \boldsymbol{A}^r_{1,*}) \in \mathcal{K}^0_\sigma \times \mathcal{T} \times \mathbb{A}_1$ such that

$$ {}^t\boldsymbol{K} : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \nabla f(\boldsymbol{\alpha}_*) - \nabla h(\boldsymbol{\alpha}_*) - \begin{pmatrix} \boldsymbol{A}^r_{1,*} \\ 0 \end{pmatrix} \in \mathcal{C} \tag{12} $$

for all $t \geq T$. Let $(\boldsymbol{\rho}, \boldsymbol{\alpha}, \boldsymbol{A}^d, \boldsymbol{A}^r)$ be an arbitrary solution of the evolution problem (10) and define

$$ W(t) = \int_\Omega \frac{1}{2}(\boldsymbol{\rho} - \frac{\boldsymbol{\rho}_*}{m}) : \boldsymbol{L}^{-1} : (\boldsymbol{\rho} - \frac{\boldsymbol{\rho}_*}{m}) + f(\boldsymbol{\alpha}) + h(\boldsymbol{\alpha})\, d\boldsymbol{x}. \tag{13} $$

Since $f \geq 0$, $h \geq 0$ and $\boldsymbol{L}$ is a positive definite tensor, the function W is positive for all t. For $t \geq T$ we have

$$ \dot{W}(t) = \int_\Omega [(\boldsymbol{\rho} - \frac{\boldsymbol{\rho}_*}{m}) : \boldsymbol{L}^{-1} : \dot{\boldsymbol{\rho}} + (\nabla f(\boldsymbol{\alpha}) + \nabla h(\boldsymbol{\alpha})).\dot{\boldsymbol{\alpha}}]\, d\boldsymbol{x} $$

where the distinctive property $\dot{\boldsymbol{\rho}}_* = 0$ has been used.

Following a reasoning similar to [18] leads to the inequality

$$ \begin{aligned} (m-1)\int_T^t D(t)\, dt \leq \;& mW(T) + \int_\Omega [-f(\boldsymbol{\alpha}_*) + \nabla f(\boldsymbol{\alpha}_*).(\boldsymbol{\alpha}_* - \boldsymbol{\alpha}(T))]\, d\boldsymbol{x} \\ & + \int_\Omega (\boldsymbol{A}^r_* + \nabla h(\boldsymbol{\alpha}_*)).(\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(T))\, d\boldsymbol{x} \end{aligned} \tag{14} $$

where $\boldsymbol{A}^r_* = (\boldsymbol{A}^r_{1,*}, 0)$. Now observe that the very last term in (14) can be bounded independently on time t. Since $\mathcal{T}_1$ is bounded, there indeed exists a constant $K > 0$ such that $\|\boldsymbol{\alpha}_1\| \leq K$ for all $\boldsymbol{\alpha}_1 \in \mathcal{T}_1$. We have

$$ \begin{aligned} \|(\boldsymbol{A}^r_* + \nabla h(\boldsymbol{\alpha}_*)).(\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(T))\| &= \|(\boldsymbol{A}^r_{1,*} + h'(\boldsymbol{\alpha}_{1,*})).(\boldsymbol{\alpha}_1(t) - \boldsymbol{\alpha}_1(T))\| \\ &\leq \|\boldsymbol{A}^r_{1,*} + h'(\boldsymbol{\alpha}_{1,*})\|.\|\boldsymbol{\alpha}_1(t) - \boldsymbol{\alpha}_1(T)\| \\ &\leq 2K\|\boldsymbol{A}^r_{1,*} + h'(\boldsymbol{\alpha}_{1,*})\| \end{aligned} $$

Therefore

$$ \begin{aligned} (m-1)\int_T^t D(t)\, dt \leq \;& mW(T) + \int_\Omega [-f(\boldsymbol{\alpha}_*) + \nabla f(\boldsymbol{\alpha}_*).(\boldsymbol{\alpha}_* - \boldsymbol{\alpha}(T))]\, d\boldsymbol{x} \\ & + 2K \int_\Omega \|\boldsymbol{A}^r_* + \nabla h(\boldsymbol{\alpha}_*)\|\, d\boldsymbol{x} \end{aligned} \tag{15} $$

The right-hand side of that last inequality is independent on t and therefore $\int_T^t D(t)\, dt$ is bounded as $t \to +\infty$. The condition (12) thus gives a sufficient condition for shakedown to occur. Since the field $\boldsymbol{A}^r_{1,*}$ in (12) is free from any constraint, Eq.(12) is equivalent to

$$ \begin{aligned} & \boldsymbol{\rho}_* \in \mathcal{K}^0_\sigma, \boldsymbol{\alpha}_* \in \mathcal{T}, \boldsymbol{B}_1 \in \mathbb{A}_1, \\ & \begin{pmatrix} {}^t\boldsymbol{K}_1 : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B}_1 \\ {}^t\boldsymbol{K}_2 : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - f_{,2}(\boldsymbol{\alpha}_*) \end{pmatrix} \in \mathcal{C} \end{aligned} \tag{16} $$

Let $\mathbb{B}_2 = \{f_{,2}(\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathcal{T}\}$. Eq. (16) can equivalently be rewritten as

$$\begin{array}{l} \boldsymbol{\rho}_* \in \mathcal{K}^0_\sigma, \boldsymbol{B}_1 \in \mathbb{A}_1, \boldsymbol{B}_2 \in \mathbb{B}_2, \\ \left(\begin{array}{l} {}^t\boldsymbol{K}_1 : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B}_1 \\ {}^t\boldsymbol{K}_2 : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B}_2 \end{array} \right) \in \mathcal{C} \end{array} \tag{17}$$

Using the notation $\boldsymbol{B} = (\boldsymbol{B}_1, \boldsymbol{B}_2)$, Eq. (17) becomes

$$\begin{array}{l} \boldsymbol{\rho}_* \in \mathcal{K}^0_\sigma, \boldsymbol{B} \in \mathbb{A}_1 \times \mathbb{B}_2, \\ {}^t\boldsymbol{K} : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B} \in \mathcal{C} \end{array} \tag{18}$$

We can thus state the following theorem:

Static shakedown theorem. *If there exists $m > 1$, $T \geq 0$ and time-independent fields $(\boldsymbol{\rho}_*, \boldsymbol{B})$ such that Eq.(18) is satisfied for all $t \geq T$, then there is shakedown, whatever the initial condition is.*

There is a simple geometric interpretation of that theorem: consider the curve $\Gamma(t)$ described by ${}^t\boldsymbol{K} : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*)$ in the space $\mathbb{A}_1 \times \mathbb{A}_2$. Shakedown occurs if, up to a translation in $\mathbb{A}_1 \times \boldsymbol{B}_2$, the curve Γ is enclosed in the elasticity domain $\mathcal{C}$.

As an example, consider the material model in [2] as briefly described in section 2. Using the presented theorem, it can be easily be seen that shakedown occurs if $\|m\boldsymbol{s}^E(t) - \boldsymbol{B}_1\| \leq R$ where $\boldsymbol{s}^E$ is the deviatoric part of $\boldsymbol{\sigma}^E$. The obtained shakedown condition thus reduces to a restriction on the diameter of the curve $\boldsymbol{s}^E(t)$, as for shakedown in linear kinematic hardening plasticity [13, 15].

Observe that we did not assume the convexity of h. This is a welcome feature for the shakedown analysis of SMA structures as the function h associated with some micromechanical SMA models is not necessarily convex [7, 17, 19, 20].

For simplicity, from here onward we restrict our attention to cyclic loadings: the function $\boldsymbol{\sigma}^E$ is assumed to be periodic in time with a period T. The static shakedown theorem motivates the definition of the static safety coefficient m_S by

$$m_S = \sup\{m : \exists(\boldsymbol{\rho}_*, \boldsymbol{B}) \text{ verifying (18) for all } 0 \leq t \leq T\} \tag{19}$$

In practice, one may select particular values of $(\boldsymbol{\rho}_*, \boldsymbol{B})$ (possibly through numerical procedure), which leads to lower bounds on m_S. Upper bounds on m_S can be obtained by a kinematic shakedown theorem, as presented in the next section.

5 KINEMATIC SHAKEDOWN THEOREM

Consider $m > 0$ such that (18) is satisfied for $t \in [0, T]$ by some time-independent fields $(\boldsymbol{\rho}_*, \boldsymbol{B})$. Let $\boldsymbol{\alpha}(t) : [0, T] \mapsto \mathcal{T}$ be such that

$$\boldsymbol{\alpha}_1(0) = \boldsymbol{\alpha}_1(T) \ , \ \boldsymbol{K}_2.(\boldsymbol{\alpha}_2(T) - \boldsymbol{\alpha}_2(0)) \in \mathcal{K}^0_\epsilon. \tag{20}$$

Let

$$P(\dot{\boldsymbol{\alpha}}) = \sup_{\boldsymbol{A}\in\mathcal{C}} \boldsymbol{A}.\dot{\boldsymbol{\alpha}}.$$

Since ${}^t\boldsymbol{K} : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B} \in \mathcal{C}$, we have

$$({}^t\boldsymbol{K} : (m\boldsymbol{\sigma}^E + \boldsymbol{\rho}_*) - \boldsymbol{B}).\dot{\boldsymbol{\alpha}} \leq P(\dot{\boldsymbol{\alpha}}).$$

Integrating over the domain Ω and over the time interval $[0, T]$, we obtain, omitting the details of the calculations,

$$m \leq \frac{\displaystyle\int_0^T \int_\Omega P(\dot{\boldsymbol{\alpha}}) d\boldsymbol{x} dt + M \int_\Omega \|\boldsymbol{\alpha}_2(T) - \boldsymbol{\alpha}_2(0)\| d\boldsymbol{x}}{\displaystyle\int_0^T \int_\Omega \boldsymbol{\sigma}^E : \boldsymbol{K}.\dot{\boldsymbol{\alpha}} d\boldsymbol{x} dt}$$

where $M = \sup\{\|\boldsymbol{B}_2\| : \boldsymbol{B}_2 \in \mathbb{B}_2\}$. From the definition (19) we can thus formulate the following theorem:

Kinematic shakedown theorem. *We have $m_S \leq m_K$ where m_K is the kinematic safety coefficient defined by*

$$m_K = \inf\left\{\frac{\displaystyle\int_0^T \int_\Omega P(\dot{\boldsymbol{\alpha}}) d\boldsymbol{x} dt + M \int_\Omega .\|\boldsymbol{\alpha}_2(T) - \boldsymbol{\alpha}_2(0)\| d\boldsymbol{x}}{\displaystyle\int_0^T \int_\Omega \boldsymbol{\sigma}^E : \boldsymbol{K}.\dot{\boldsymbol{\alpha}} d\boldsymbol{x} dt} : \boldsymbol{\alpha}(t) \text{ verifying } (20)\right\}$$

In practice, selecting special fields $\boldsymbol{\alpha}(t)$ gives an upper bound on m_K and consequently an upper bound on m_S.

If $\boldsymbol{B}_2$ is unbounded, i.e $M = \infty$, then one should only consider fields $\boldsymbol{\alpha}(t)$ verifying that $\boldsymbol{\alpha}_i(0) = \boldsymbol{\alpha}_i(T)$ for $i = 1, 2$. In particular, for the model in [2], the safety coefficient m_K becomes

$$m_K = \inf\left\{\frac{\displaystyle\int_0^T \int_\Omega P(\dot{\boldsymbol{\alpha}}) d\boldsymbol{x} dt}{\displaystyle\int_0^T \int_\Omega \boldsymbol{\sigma}^E : \dot{\boldsymbol{\alpha}}_1 d\boldsymbol{x} dt} : \boldsymbol{\alpha}(t) \text{ verifying } \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(T)\right\}$$

6 CONCLUDING REMARKS

In this paper, we have presented a static and a kinematic shakedown theorems for SMA models coupling phase-transformation with permanent inelasticity. We emphasize that those theorems are path-independent: the obtained shakedown conditions do not depend on the initial state of the system (which for instance would correspond to some initial residual stress). Since there is an established connection between fatigue and energy dissipation, the proposed theorems could possibly be useful for the fatigue design of SMA systems. Further investigation is required to clarify that point.

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